

Symmetries for the Ablowitz-Ladik hierarchy: II. Integrable discrete nonlinear Schrödinger equation and discrete AKNS hierarchy

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Abstract

In the paper we continue to consider symmetries related to the Ablowitz-Ladik hierarchy. We derive symmetries for the integrable discrete nonlinear Schrödinger hierarchy and discrete AKNS hierarchy. The integrable discrete nonlinear Schrödinger hierarchy are in scalar form and its two sets of symmetries are shown to form a Lie algebra. We also present discrete AKNS isospectral flows, non-isospectral flows and their recursion operator. In continuous limit these flows go to the continuous AKNS flows and the recursion operator goes to the square of the AKNS recursion operator. These discrete AKNS flows form a Lie algebra which plays a key role in constructing symmetries and their algebraic structures for both the integrable discrete nonlinear Schrödinger hierarchy and discrete AKNS hierarchy. Structures of the obtained algebras are different structures from those in continuous cases which usually are centerless Kac-Moody-Virasoro type. These algebra deformations are explained through continuous limit and *degree* in terms of lattice spacing parameter h .

Key words: Ablowitz-Ladik hierarchies; symmetries; discrete nonlinear Schrödinger equation; discrete AKNS hierarchy; algebra deformation.

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1 Introduction

In the previous paper [1], to which we refer as Part I, we have derived symmetries and their algebras for the isospectral and non-isospectral four-potential Ablowitz-Ladik (AL) hierarchies, and these results including symmetries of hierarchies were shown to be reduced to the two-potential case.

Among the equations related to the two-potential AL spectral problem, one of the most physically meaningful systems is the integrable discrete nonlinear Schrödinger (IDNLS) equation [2–4]:

$$iQ_{n,t} = Q_{n+1} + Q_{n-1} - 2Q_n + \varepsilon|Q_n|^2(Q_{n+1} + Q_{n-1}), \quad (1.1)$$

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where $\varepsilon = \pm 1$ and i is the imaginary unit. This equation differs from the discrete NLS equation

$$iQ_{n,t} = Q_{n+1} + Q_{n-1} - 2Q_n + 2\varepsilon Q_n |Q_n|^2 \quad (1.2)$$

by the nonlinear term, which is not integrable but arose in many important physical contexts (cf. the introduction chapter of [4] and the references therein).

With regard to symmetries, Refs. [5, 6] derived point and generalized symmetries for the IDNLS equation (1.1). Two subalgebras were obtained [5] and then a general structure for the whole symmetry algebra was described [6]. In the present paper we hope to go further than Refs. [5, 6]. In fact, Eq.(1.1) consists of positive as well as negative order AL isospectral flows, which corresponds to a central-difference discretization for the second order derivative in the continuous NLS equation. Since in Part I and also in [7] we have derived algebraic relations for arbitrary two flows among the positive and negative-order, isospectral and non-isospectral, hierarchies related to the two-potential AL spectral problem, it is possible to get infinitely many symmetries with a clear algebraic structure for the IDNLS equation (1.1).

Our plan is the following. We will start from the isospectral and non-isospectral flows of the two-potential AL system, which serve as basic flows. By suitable linear combinations we can get isospectral and non-isospectral IDNLS flows separately (cf. [5, 8] where a uniformed hierarchy was given). Since the IDNLS equation (1.1) is in scalar form, we then try to derive algebraic structure of the scalar IDNLS flows. By the obtained structure we can present infinitely many symmetries which form a Lie algebra with clear structure. Going further we can get similar results for the IDNLS hierarchy. As in the continuous AKNS system, the IDNLS flows can be a reduction of some flows which we call the discrete AKNS (DAKNS) flows in the paper. We find these flows and their recursion operator correspond to the continuous AKNS flows and the square of the AKNS recursion operator. In our paper the algebraic structure of the DAKNS flows will play a key role in deriving symmetries and their algebras for the IDNLS hierarchy and the DAKNS hierarchy. Finally, by comparison one can see the difference between the obtained algebras and their continuous counterparts. These algebra deformations will also be explained by considering continuous limit and introducing *degree* of discrete elements.

The paper is organized as follows. Sec.2 contains basic results of the two-potential AL system, including flows, recursion operator and algebraic structure. In Sec.3 we present isospectral and non-isospectral IDNLS flows and their algebraic structures. Sec.4 derives symmetries and Lie algebras for the IDNLS equation and hierarchy. Sec.5 discusses the DAKNS flows and continuous limits. In Sec.6 algebra deformations from discrete case to continuous case are listed out and explanation follows. Finally, Sec.7 gives conclusions.

2 Flows related to the AL spectral problem

Let us list out the main results of Ref. [7] as the starting point. We will also follow the notations used in [7] without any confusion with Part I. The two-potential AL spectral problem (also called discrete Zakharov-Shabat spectral problem) is

$$\Phi_{n+1} = M\Phi_n, \quad M = \begin{pmatrix} \lambda & Q_n \\ R_n & \frac{1}{\lambda} \end{pmatrix}, \quad u_n = \begin{pmatrix} Q_n \\ R_n \end{pmatrix}, \quad \Phi_n = \begin{pmatrix} \phi_{1,n} \\ \phi_{2,n} \end{pmatrix}. \quad (2.1)$$

The isospectral AL hierarchy and non-isospectral AL hierarchy are respectively

$$u_{n,t} = K^{(l)} = L^l K^{(0)}, \quad K^{(0)} = (Q_n, -R_n)^T, \quad l \in \mathbb{Z}, \quad (2.2a)$$

$$u_{n,t} = \sigma^{(l)} = L^l \sigma^{(0)}, \quad \sigma^{(0)} = (2n+1)(Q_n, -R_n)^T, \quad l \in \mathbb{Z}, \quad (2.2b)$$

where L is the recursion operator defined as

$$L = \begin{pmatrix} E & 0 \\ 0 & E^{-1} \end{pmatrix} + \begin{pmatrix} -Q_n E \\ R_n \end{pmatrix} (E-1)^{-1} (R_n E, Q_n E^{-1}) + \gamma_n^2 \begin{pmatrix} -E Q_n \\ R_{n-1} \end{pmatrix} (E-1)^{-1} (R_n, Q_n) \frac{1}{\gamma_n^2}, \quad (2.3)$$

$\gamma_n = \sqrt{1 - Q_n R_n}$ and E is a shift operator defined as $E^j f(n) = f(n+j)$, $\forall j \in \mathbb{Z}$. L is invertible with

$$L^{-1} = \begin{pmatrix} E^{-1} & 0 \\ 0 & E \end{pmatrix} + \begin{pmatrix} Q_n \\ -R_n E \end{pmatrix} (E-1)^{-1} (R_n E^{-1}, Q_n E) + \gamma_n^2 \begin{pmatrix} Q_{n-1} \\ -E R_n \end{pmatrix} (E-1)^{-1} (R_n, Q_n) \frac{1}{\gamma_n^2}.$$

The isospectral flows $\{K^{(l)}\}$ and non-isospectral flows $\{\sigma^{(l)}\}$ form a centreless Kac-Moody-Virasoro (KMV) algebra, $\forall l, s \in \mathbb{Z}$,

$$[[K^{(l)}, K^{(s)}]] = 0, \quad (2.4a)$$

$$[[K^{(l)}, \sigma^{(s)}]] = 2l K^{(l+s)}, \quad (2.4b)$$

$$[[\sigma^{(l)}, \sigma^{(s)}]] = 2(l-s) \sigma^{(l+s)}, \quad (2.4c)$$

where the Lie product $[[\cdot, \cdot]]$ is defined as in Part I, i.e.,

$$[[f, g]] = f'[g] - g'[f], \quad (2.5)$$

in which $f = (f_1(u_n), f_2(u_n))^T$, $g = (g_1(u_n), g_2(u_n))^T$, $f'[g]$ is the Gateaux derivative of f w.r.t u_n in direction g , i.e.,

$$f'[g] = f(u_n)'[g] = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} f(u_n + \epsilon g), \quad \epsilon \in \mathbb{R}, \quad (2.6)$$

and vice versa for $g'[f]$. (2.4) is a generalization of the results in [9].

3 Isospectral and non-isospectral IDNLS hierarchies

3.1 Hierarchies

To derive IDNLS hierarchies, let us introduce auxiliary flows

$$K_{[0]}^{(0)} = K^{(0)}, \quad (3.1a)$$

$$K_{[1]}^{(0)} = \frac{1}{2}(L - L^{-1})K^{(0)} = \frac{1}{2} \begin{pmatrix} (1 - Q_n R_n)(Q_{n+1} - Q_{n-1}) \\ (1 - Q_n R_n)(R_{n+1} - R_{n-1}) \end{pmatrix}, \quad (3.1b)$$

$$\sigma_{[0]}^{(0)} = \frac{1}{2}\sigma^{(0)}, \quad (3.2a)$$

$$\begin{aligned} \sigma_{[1]}^{(0)} = \frac{1}{4}(L - L^{-1})\sigma^{(0)} = \frac{1}{4} \begin{pmatrix} (1 - Q_n R_n)[(2n+3)Q_{n+1} - (2n-1)Q_{n-1}] \\ (1 - Q_n R_n)[(2n+3)R_{n+1} - (2n-1)R_{n-1}] \end{pmatrix} \\ - \frac{1}{2} \begin{pmatrix} Q_n \\ -R_n \end{pmatrix} (E-1)^{-1}(Q_{n+1}R_n - Q_n R_{n+1}), \end{aligned} \quad (3.2b)$$

and operator

$$\begin{aligned} \mathcal{L} &= L - 2I + L^{-1} \\ &= \begin{pmatrix} E + E^{-1} - 2 & 0 \\ 0 & E + E^{-1} - 2 \end{pmatrix} + \gamma_n^2 \begin{pmatrix} Q_{n-1} - Q_{n+1}E \\ R_{n-1} - R_{n+1}E \end{pmatrix} (E-1)^{-1}(R_n, Q_n) \frac{1}{\gamma_n^2} \\ &\quad + \begin{pmatrix} Q_n \\ -R_n E \end{pmatrix} (E-1)^{-1}(R_n E^{-1}, Q_n E) - \begin{pmatrix} Q_n E \\ -R_n \end{pmatrix} (E-1)^{-1}(R_n E, Q_n E^{-1}), \end{aligned} \quad (3.3)$$

where I is the 2×2 unit matrix. By $*$ we denote complex conjugate. Then we define

$$\tilde{K}_{[0]}^{(0)} = K_{[0]}^{(0)} \Big|_{R_n = -\varepsilon Q_n^*} = (Q_n, \varepsilon Q_n^*)^T, \quad (3.4a)$$

$$\tilde{K}_{[1]}^{(0)} = K_{[1]}^{(0)} \Big|_{R_n = -\varepsilon Q_n^*} = \frac{1}{2} \begin{pmatrix} (1 + \varepsilon |Q_n|^2)(Q_{n+1} - Q_{n-1}) \\ -\varepsilon(1 + \varepsilon |Q_n|^2)(Q_{n+1}^* - Q_{n-1}^*) \end{pmatrix}, \quad (3.4b)$$

$$\tilde{\sigma}_{[0]}^{(0)} = \sigma_{[0]}^{(0)} \Big|_{R_n = -\varepsilon Q_n^*} = (n + \frac{1}{2})(Q_n, \varepsilon Q_n^*)^T, \quad (3.5a)$$

$$\begin{aligned} \tilde{\sigma}_{[1]}^{(0)} = \sigma_{[1]}^{(0)} \Big|_{R_n = -\varepsilon Q_n^*} &= \frac{1}{4} \begin{pmatrix} (1 + \varepsilon |Q_n|^2)[(2n+3)Q_{n+1} - (2n-1)Q_{n-1}] \\ -\varepsilon(1 + \varepsilon |Q_n|^2)[(2n+3)Q_{n+1}^* - (2n-1)Q_{n-1}^*] \end{pmatrix} \\ &\quad + \frac{1}{2}\varepsilon \begin{pmatrix} Q_n \\ \varepsilon Q_n^* \end{pmatrix} (E-1)^{-1}(Q_{n+1}Q_n^* - Q_n Q_{n+1}^*), \end{aligned} \quad (3.5b)$$

the operator

$$\tilde{\mathcal{L}} = \mathcal{L} \Big|_{R_n = -\varepsilon Q_n^*}, \quad (3.6)$$

and further define the flows

$$\tilde{K}_{[j]}^{(l)} = \tilde{\mathcal{L}}^l \tilde{K}_{[j]}^{(0)}, \quad j \in \{0, 1\}, \quad (3.7a)$$

$$\tilde{\sigma}_{[j]}^{(s)} = \tilde{\mathcal{L}}^s \tilde{\sigma}_{[j]}^{(0)}, \quad j \in \{0, 1\}. \quad (3.7b)$$

To make clear the relationship of two components in $\tilde{K}_{[j]}^{(l)}$ and $\tilde{\sigma}_{[j]}^{(s)}$, we introduce function sets

$$\mathcal{A}_{[j]}(\varepsilon) = \{(f_1, (-1)^j \varepsilon f_1^*)^T\}, \quad j \in \{0, 1\},$$

where f_1 is an arbitrary scalar function. Then one can find that

$$\tilde{K}_{[j]}^{(0)}, \tilde{\sigma}_{[j]}^{(0)} \in \mathcal{A}_{[j]}(\varepsilon), \quad (3.8)$$

and $\tilde{\mathcal{L}}$ provides a self-transformation for both $\mathcal{A}_{[0]}(\varepsilon)$ and $\mathcal{A}_{[1]}(\varepsilon)$, i.e., $\forall \alpha \in \mathcal{A}_{[0]}(\varepsilon)$, $\tilde{\mathcal{L}}\alpha \in \mathcal{A}_{[0]}(\varepsilon)$, and $\forall \beta \in \mathcal{A}_{[1]}(\varepsilon)$, $\tilde{\mathcal{L}}\beta \in \mathcal{A}_{[1]}(\varepsilon)$. That means

$$\tilde{K}_{[j]}^{(l)}, \tilde{\sigma}_{[j]}^{(s)} \in \mathcal{A}_{[j]}(\varepsilon). \quad (3.9)$$

Noting that $\tilde{u}_n = u_n|_{R_n = -\varepsilon Q_n^*} = (Q_n, -\varepsilon Q_n^*)^T \in \mathcal{A}_{[1]}(\varepsilon)$, we then define the following isospectral IDNLS hierarchy

$$i^{1-j} \tilde{u}_{n,t_m,j} = \tilde{K}_{[j]}^{(m)} = \tilde{\mathcal{L}}^m \tilde{K}_{[j]}^{(0)}, \quad m = 0, 1, 2, \dots, \quad (3.10a)$$

and non-isospectral IDNLS hierarchy

$$i^{1-j} \tilde{u}_{n,t_s,j} = \tilde{\sigma}_{[j]}^{(s)} = \tilde{\mathcal{L}}^s \tilde{\sigma}_{[j]}^{(0)}, \quad s = 0, 1, 2, \dots, \quad (3.10b)$$

where $j \in \{0, 1\}$, $\tilde{\mathcal{L}}$ is the recursion operator, and we add subindexes m, j and s, j for t as labels of equations in their hierarchy. In fact, the IDNLS equation (1.1) comes from (3.10a) with $j = 0, m = 1$, i.e.,

$$i \tilde{u}_{n,t_1,0} = i \begin{pmatrix} Q_n \\ -\varepsilon Q_n^* \end{pmatrix}_{t_{1,0}} = \tilde{K}_{[0]}^{(1)} = \begin{pmatrix} Q_{n+1} + Q_{n-1} - 2Q_n + \varepsilon Q_n Q_n^* (Q_{n+1} + Q_{n-1}) \\ \varepsilon Q_{n+1}^* + \varepsilon Q_{n-1}^* - 2\varepsilon Q_n^* + Q_n Q_n^* (Q_{n+1}^* + Q_{n-1}^*) \end{pmatrix}. \quad (3.11)$$

In non-isospectral case, (3.10b) with $j = 0$ and $s = 1$ reads

$$\begin{aligned} i \begin{pmatrix} Q_n \\ -\varepsilon Q_n^* \end{pmatrix}_{t_{1,0}} &= \tilde{\sigma}_{[0]}^{(1)} = \frac{1}{2} \begin{pmatrix} (1 + \varepsilon Q_n Q_n^*)[(2n+3)Q_{n+1} + (2n-1)Q_{n-1}] - 2(2n+1)Q_n \\ \varepsilon(1 + \varepsilon Q_n Q_n^*)[(2n+3)Q_{n+1}^* + (2n-1)Q_{n-1}^*] - 2\varepsilon(2n+1)Q_n^* \end{pmatrix} \\ &\quad + \varepsilon \begin{pmatrix} Q_n \\ \varepsilon Q_n^* \end{pmatrix} (E-1)^{-1} (Q_{n+1} Q_n^* + Q_n Q_{n+1}^*), \end{aligned} \quad (3.12)$$

of which the first row provides a non-isospectral IDNLS equation

$$\begin{aligned} i Q_{n,t_1,0} &= \frac{1}{2} (1 + \varepsilon Q_n Q_n^*) [(2n+3)Q_{n+1} + (2n-1)Q_{n-1}] - (2n+1)Q_n \\ &\quad + \varepsilon Q_n (E-1)^{-1} (Q_{n+1} Q_n^* + Q_n Q_{n+1}^*), \end{aligned} \quad (3.13)$$

which goes to a non-isospectral NLS equation in continuous limit (see Sec.5).

The flows $\{i^{j-1} \tilde{K}_{[j]}^{(l)}\}$ and $\{i^{j-1} \tilde{\sigma}_{[j]}^{(s)}\}$ defined in (3.7) are called isospectral and non-isospectral IDNLS flows in vector form, respectively. We add multiplier i^{j-1} so that they are always in the set $\mathcal{A}_{[j]}(\varepsilon)$ to which \tilde{u}_n belongs. The first components of $\{i^{j-1} \tilde{K}_{[j]}^{(l)}\}$ and $\{i^{j-1} \tilde{\sigma}_{[j]}^{(s)}\}$, which we respectively denote them by $\{i^{j-1} \tilde{K}_{[j,1]}^{(l)}\}$ and $\{i^{j-1} \tilde{\sigma}_{[j,1]}^{(s)}\}$, are called isospectral and non-isospectral IDNLS flows in scalar form. Thus the scalar form of isospectral and non-isospectral IDNLS hierarchies can be written as

$$i^{1-j} Q_{n,t_m,j} = \tilde{K}_{[j,1]}^{(m)}, \quad m = 0, 1, 2, \dots, \quad (3.14a)$$

$$i^{1-j} Q_{n,t_s,j} = \tilde{\sigma}_{[j,1]}^{(s)}, \quad s = 0, 1, 2, \dots. \quad (3.14b)$$

3.2 Algebraic structures of the INDLS flows

Making use of (2.4) one can derive the algebraic relations for the IDNLS flows.

Theorem 3.1. *Suppose that Q_n is the only independent variable and the Gateaux derivative is defined w.r.t. Q_n . Then the scalar isospectral and non-isospectral IDNLS flows $\{i^{j-1}\tilde{K}_{[j],1}^{(m)}\}$ and $\{i^{j-1}\tilde{\sigma}_{[j],1}^{(s)}\}$ form a Lie algebra through the Lie product $[\![\cdot, \cdot]\!]_{Q_n}$ with the following structure*

$$[\![i^{j-1}\tilde{K}_{[j],1}^{(m)}, i^{k-1}\tilde{K}_{[k],1}^{(s)}]\!]_{Q_n} = 0, \quad (3.15a)$$

$$[\![i\tilde{K}_{[0],1}^{(m)}, -i\tilde{\sigma}_{[0],1}^{(s)}]\!]_{Q_n} = -2m\tilde{K}_{[1],1}^{(m+s-1)}, \quad (3.15b)$$

$$[\![i\tilde{K}_{[0],1}^{(m)}, \tilde{\sigma}_{[1],1}^{(s)}]\!]_{Q_n} = -\frac{1}{2}im(\tilde{K}_{[0],1}^{(m+s+1)} + 4\tilde{K}_{[0],1}^{(m+s)}), \quad (3.15c)$$

$$[\![\tilde{K}_{[1],1}^{(m)}, i^{j-1}\tilde{\sigma}_{[j],1}^{(s)}]\!]_{Q_n} = \frac{1}{2}i^{j-1}[(m+1)\tilde{K}_{[j],1}^{(m+s+1)} + 2(2m+1)\tilde{K}_{[j],1}^{(m+s)}], \quad (3.15d)$$

$$[\![i\tilde{\sigma}_{[0],1}^{(m)}, -i\tilde{\sigma}_{[0],1}^{(s)}]\!]_{Q_n} = -2(m-s)\tilde{\sigma}_{[1],1}^{(m+s-1)}, \quad (3.15e)$$

$$[\![i^{j-1}\tilde{\sigma}_{[j],1}^{(m)}, \tilde{\sigma}_{[1],1}^{(s)}]\!]_{Q_n} = \frac{1}{2}i^{j-1}[(m-s-1+j)\tilde{\sigma}_{[j],1}^{(m+s+1)} + 2(2m-2s-1+j)\tilde{\sigma}_{[j],1}^{(m+s)}], \quad (3.15f)$$

where $j, k \in \{0, 1\}$, $m, s \geq 0$ and we set $\tilde{K}_{[j],1}^{(-1)} = \tilde{\sigma}_{[j],1}^{(-1)} = 0$ once they appear on the r.h.s. of (3.15). We note that hereafter by $[\![\cdot, \cdot]\!]_{Q_n}$ we denote the product defined through the Gateaux derivative w.r.t. Q_n .

We prove the theorem by two steps. First, we derive algebraic structures for the following vector flows

$$K_{[j]}^{(l)} = \mathcal{L}^l K_{[j]}^{(0)}, \quad \sigma_{[j]}^{(l)} = \mathcal{L}^l \sigma_{[j]}^{(0)}, \quad j \in \{0, 1\}, \quad l = 0, 1, \dots, \quad (3.16)$$

where $K_{[j]}^{(0)}$ and $\sigma_{[j]}^{(0)}$ are given in (3.1) and (3.2), respectively. $\{K_{[j]}^{(l)}\}$ and $\{\sigma_{[j]}^{(l)}\}$ are called (semi-)DAKNS flows (see Sec.5.1) ¹. For these flows we have

Lemma 3.1. *The flows $\{K_{[j]}^{(m)}\}$ and $\{\sigma_{[j]}^{(s)}\}$ form a Lie algebra, denoted by \mathcal{D} , through $[\![\cdot, \cdot]\!]_{Q_n}$ with structure*

$$[\![K_{[j]}^{(m)}, K_{[k]}^{(s)}]\!] = 0, \quad (3.17a)$$

$$[\![K_{[0]}^{(m)}, \sigma_{[0]}^{(s)}]\!] = 2mK_{[1]}^{(m+s-1)}, \quad (3.17b)$$

$$[\![K_{[0]}^{(m)}, \sigma_{[1]}^{(s)}]\!] = \frac{1}{2}m(K_{[0]}^{(m+s+1)} + 4K_{[0]}^{(m+s)}), \quad (3.17c)$$

$$[\![K_{[1]}^{(m)}, \sigma_{[j]}^{(s)}]\!] = \frac{1}{2}[(m+1)K_{[j]}^{(m+s+1)} + 2(2m+1)K_{[j]}^{(m+s)}], \quad (3.17d)$$

$$[\![\sigma_{[0]}^{(m)}, \sigma_{[0]}^{(s)}]\!] = 2(m-s)\sigma_{[1]}^{(m+s-1)}, \quad (3.17e)$$

$$[\![\sigma_{[j]}^{(m)}, \sigma_{[1]}^{(s)}]\!] = \frac{1}{2}[(m-s-1+j)\sigma_{[j]}^{(m+s+1)} + 2(2m-2s-1+j)\sigma_{[j]}^{(m+s)}], \quad (3.17f)$$

¹In continuous limit one can find $K_{[0]}^{(0)} \sim (q, -r)^T$, $K_{[1]}^{(0)} \sim (q_x, r_x)^T$, $\sigma_{[0]}^{(0)} \sim (xq, -xr)^T$, $\sigma_{[1]}^{(0)} \sim (q + xq_x, r + xr_x)^T$ and $\mathcal{L} \sim L_{AKNS}^2$ where L_{AKNS} is the recursion operator of the AKNS system.

where $j, k \in \{0, 1\}$, the Gateaux derivative is still defined w.r.t. u_n , $m, s \geq 0$ and we set $K_{[j]}^{(-1)} = \sigma_{[j]}^{(-1)} = 0$ once they appear on the r.h.s. of (3.17).

Proof. We only prove (3.17c). The others can be proved similarly. Noting that

$$\mathcal{L}^m = (L - 2I + L^{-1})^m = \sum_{r=0}^m \sum_{j=0}^{m-r} C_m^r C_{m-r}^j (-2)^j L^{m-2r-j}, \quad (3.18)$$

and by this we write $K_{[0]}^{(m)}$ and $\sigma_{[1]}^{(s)}$ as

$$\begin{aligned} K_{[0]}^{(m)} &= \sum_{r=0}^m \sum_{j=0}^{m-r} C_m^r C_{m-r}^j (-2)^j K^{(m-2r-j)}, \\ \sigma_{[1]}^{(s)} &= \frac{1}{4} \sum_{h=0}^s \sum_{k=0}^{s-h} C_s^h C_{s-h}^k (-2)^k (\sigma^{(s-2h-k+1)} - \sigma^{(s-2h-k-1)}). \end{aligned}$$

Substituting them into $\llbracket K_{[0]}^{(m)}, \sigma_{[1]}^{(s)} \rrbracket$ and making using of the Lie product relation (2.4b) yield

$$\llbracket K_{[0]}^{(m)}, \sigma_{[1]}^{(s)} \rrbracket = A + B + C, \quad (3.19)$$

where

$$\begin{aligned} A &= \frac{1}{2} m (L - L^{-1}) \sum_{r=0}^m \sum_{j=0}^{m-r} C_m^r C_{m-r}^j (-2)^j \sum_{h=0}^s \sum_{k=0}^{s-h} C_s^h C_{s-h}^k (-2)^k K^{(m+s-2r-2h-j-k)}, \\ B &= -(L - L^{-1}) \sum_{r=0}^m \sum_{j=0}^{m-r} r C_m^r C_{m-r}^j (-2)^j \sum_{h=0}^s \sum_{k=0}^{s-h} C_s^h C_{s-h}^k (-2)^k K^{(m+s-2r-2h-j-k)}, \\ C &= -\frac{1}{2} (L - L^{-1}) \sum_{r=0}^m \sum_{j=0}^{m-r} j C_m^r C_{m-r}^j (-2)^j \sum_{h=0}^s \sum_{k=0}^{s-h} C_s^h C_{s-h}^k (-2)^k K^{(m+s-2r-2h-j-k)}. \end{aligned}$$

Next, still using (3.18), the first term A is nothing but

$$A = \frac{1}{2} m (L - L^{-1}) \mathcal{L}^m \mathcal{L}^s K^{(0)} = \frac{1}{2} m (L - L^{-1}) K_{[0]}^{(m+s)}.$$

For the second term B where the summation for r essentially starts from $r = 1$, again using (3.18) we have

$$B = -(L - L^{-1}) \mathcal{L}^s \sum_{r=1}^m r C_m^r (L - 2I)^{m-r} L^{-r} K^{(0)}.$$

It then follows from the formula $r C_m^r = m C_{m-1}^{r-1}$ that

$$\begin{aligned} B &= -(L - L^{-1}) \mathcal{L}^s \sum_{r=0}^{m-1} m C_{m-1}^r (L - 2I)^{m-1-r} L^{-r-1} K^{(0)} \\ &= -m (L - L^{-1}) L^{-1} \mathcal{L}^s \mathcal{L}^{m-1} K^{(0)} = -m (L - L^{-1}) L^{-1} K_{[0]}^{(m+s-1)}. \end{aligned}$$

Similarly, for the last term C , using

$$jC_m^r C_{m-r}^j = (m-r)C_m^r C_{m-r-1}^{j-1} = mC_{m-1}^r C_{m-r-1}^{j-1}$$

we have

$$C = m(L - L^{-1})K_{[0]}^{(m+s-1)}.$$

Then we have

$$\begin{aligned} \llbracket K_{[0]}^{(m)}, \sigma_{[1]}^{(s)} \rrbracket &= A + B + C \\ &= \frac{1}{2}m(L - L^{-1})(\mathcal{L} - 2L^{-1} + 2I)K_{[0]}^{(m+s-1)} \\ &= \frac{1}{2}m(L - L^{-1})^2 K_{[0]}^{(m+s-1)} \\ &= \frac{1}{2}m(\mathcal{L}^2 + 4\mathcal{L})K_{[0]}^{(m+s-1)} \\ &= \frac{1}{2}m(K_{[0]}^{(m+s+1)} + 4K_{[0]}^{(m+s)}), \end{aligned}$$

which is just (3.17c). The other relations in (3.17) can be proved in a similar way. \square

It is easy to find that the Lie algebra \mathcal{D} is generated by the following elements

$$\{K_{[0]}^{(0)}, K_{[1]}^{(0)} \text{ (or } K_{[0]}^{(1)}), \sigma_{[0]}^{(0)}, \sigma_{[1]}^{(0)} \text{ (or } \sigma_{[0]}^{(1)}), \sigma_{[1]}^{(1)}\}. \quad (3.20)$$

The second step consists of a discussion for the consistency of the reduction of (3.17) under $R_n = -\varepsilon Q_n^*$. Let us first consider 2-dimensional vector functions:

$$f(u_n) = (f_1, f_2)^T, \quad g(u_n) = (g_1, g_2)^T, \quad h(u_n) = (h_1, h_2)^T, \quad (3.21)$$

which are related by

$$\llbracket f, g \rrbracket = h. \quad (3.22)$$

We note that if Q_n and R_n are two independent variables and there is no complex operation of Q_n and R_n in f, g , then the linear relationship holds on the complex number field \mathbb{C} , i.e.,

$$f(u_n)'[ag] = af(u_n)'[g], \quad \forall a \in \mathbb{C}. \quad (3.23)$$

However, when $R_n = -\varepsilon Q_n^*$ and Q_n is considered to be the only one independent variable, in general the above linear relationship does not hold any longer unless a is real.²

For a reasonable reduction for (3.17) the problem we have to conquer is the ‘consistency’ of the product (3.22):

- First, a consistent reduction requires two components are somehow related after reduction, for example, $h_2 = -\varepsilon h_1^*$.

²For example, $f_1 = Q_n^2 + Q_n^*$, $g_1 = Q_{n,x}$, $f_1(Q_n)'[g_1] = 2Q_n Q_{n,x} + Q_{n,x}^*$ but $f_1(Q_n)'[ig_1] = i(2Q_n Q_{n,x} - Q_{n,x}^*) \neq if_1(Q_n)'[g_1]$.

- Second, when Q_n becomes the only one independent variable in stead of (Q_n, R_n) , a consistent reduction for the product (3.22)| $R_n = -\varepsilon Q_n^*$ should provide

$$\llbracket f_1, g_1 \rrbracket_{Q_n} = f_1(Q_n)'[g_1] - g_1(Q_n)'[f_1] = h_1. \quad (3.24)$$

Such consistency for the product (3.22) can be guaranteed by taking $f(\tilde{u}_n), g(\tilde{u}_n)$ and \tilde{u}_n are in the same function set, i.e.,

$$f(\tilde{u}_n), g(\tilde{u}_n) \in A_{[1]}(\varepsilon), \quad (3.25)$$

same as \tilde{u}_n .

After the above discussion for the consistency of reduction, first, we multiply $K_{[j]}^{(m)}$ and $\sigma_{[j]}^{(s)}$ on the l.h.s. of (3.17) by i^{j-1} which just guarantees $\{i^{j-1}\tilde{K}_{[j]}^{(l)}\}$ and $\{i^{j-1}\tilde{\sigma}_{[j]}^{(s)}\}$ are in the set $A_{[1]}(\varepsilon)$ to which \tilde{u}_n belongs. Then we take the reduction $R_n = -\varepsilon Q_n^*$ and following (3.24) we get the algebraic relations for those first components, which are listed in Theorem 3.1. Taking (3.17c) as an example, we first multiply $K_{[0]}^{(m)}$ by $-i$ and then rewrite (3.17c) to

$$\llbracket -iK_{[0]}^{(m)}, \sigma_{[1]}^{(s)} \rrbracket = -\frac{1}{2}im(K_{[0]}^{(m+s+1)} + 4K_{[0]}^{(m+s)}). \quad (3.26)$$

This is then ready for a consistent reduction and after taking $R_n = -\varepsilon Q_n^*$ we get (3.15c).

4 Symmetries

4.1 Symmetries for the IDNLS equation

With the algebraic relations (3.15) in hand, we can construct symmetries for the IDNLS equation (1.1), i.e., $iQ_{n,t_1,0} = \tilde{K}_{[0],1}^{(1)}$. A scalar function $\tau = \tau(Q_n)$ is a symmetry of (1.1), if

$$\tau_{t_1,0} = -i\tilde{K}_{[0],1}^{(1)}(Q_n)'[\tau], \quad (4.1)$$

which is, equivalently,

$$\frac{\tilde{\partial}\tau}{\tilde{\partial}t_{1,0}} = \llbracket -i\tilde{K}_{[0],1}^{(1)}, \tau \rrbracket_{Q_n}, \quad (4.2)$$

where $\frac{\tilde{\partial}\tau}{\tilde{\partial}t_{1,0}}$ specially denotes the derivative of τ w.r.t. $t_{1,0}$ explicitly included in τ (cf. [1]), and the Gateaux derivative in (4.1) is defined w.r.t. Q_n .

From the algebraic structures in Theorem 3.1 and the definition (4.2) we have the following symmetries for the IDNLS equation (1.1): K -symmetries $\{i^{j-1}\tilde{K}_{[j],1}^{(m)}\}$ and τ -symmetries

$$\tau_{[0,j]}^{(1,s)} = t_{1,0} \cdot \llbracket -i\tilde{K}_{[0],1}^{(1)}, i^{j-1}\tilde{\sigma}_{[j],1}^{(s)} \rrbracket_{Q_n} + i^{j-1}\tilde{\sigma}_{[j],1}^{(s)}, \quad s = 0, 1, \dots, \quad (4.3)$$

i.e.,

$$\tau_{[0,0]}^{(1,s)} = -2t_{1,0}\tilde{K}_{[1],1}^{(s)} - i\tilde{\sigma}_{[0],1}^{(s)}, \quad (4.4a)$$

$$\tau_{[0,1]}^{(1,s)} = -\frac{1}{2}it_{1,0}(\tilde{K}_{[0],1}^{(s+2)} + 4\tilde{K}_{[0],1}^{(s+1)}) + \tilde{\sigma}_{[1],1}^{(s)}. \quad (4.4b)$$

The algebraic relations in (3.15) suggest an algebra for the symmetries of the IDNLS equation (1.1). This is concluded by

Theorem 4.1. *The isospectral IDNLS equation (1.1) can have two sets of symmetries, K -symmetries $\{i^{l-1}\tilde{K}_{[l,1]}^{(m)}\}$ and τ -symmetries $\tau_{[0,l]}^{(1,s)}$ given in (4.4), which form a Lie algebra with structure*

$$\llbracket i^{z-1}\tilde{K}_{[z,1]}^{(m)}, i^{l-1}\tilde{K}_{[l,1]}^{(s)} \rrbracket_{Q_n} = 0, \quad (4.5a)$$

$$\llbracket -i\tilde{K}_{[0,1]}^{(m)}, \tau_{[0,0]}^{(1,s)} \rrbracket_{Q_n} = -2m\tilde{K}_{[1,1]}^{(m+s-1)}, \quad (4.5b)$$

$$\llbracket -i\tilde{K}_{[0,1]}^{(m)}, \tau_{[0,1]}^{(1,s)} \rrbracket_{Q_n} = -\frac{1}{2}im(\tilde{K}_{[0,1]}^{(m+s+1)} + 4\tilde{K}_{[0,1]}^{(m+s)}), \quad (4.5c)$$

$$\llbracket \tilde{K}_{[1,1]}^{(m)}, \tau_{[0,l]}^{(1,s)} \rrbracket_{Q_n} = \frac{1}{2}i^{l-1}[(m+1)\tilde{K}_{[l,1]}^{(m+s+1)} + 2(2m+1)\tilde{K}_{[l,1]}^{(m+s)}], \quad (4.5d)$$

$$\llbracket \tau_{[0,0]}^{(1,m)}, \tau_{[0,0]}^{(1,s)} \rrbracket_{Q_n} = -2(m-s)\tau_{[0,1]}^{(1,m+s-1)}, \quad (4.5e)$$

$$\llbracket \tau_{[0,l]}^{(1,m)}, \tau_{[0,1]}^{(1,s)} \rrbracket_{Q_n} = \frac{1}{2}[(m-s-1+l)\tau_{[0,l]}^{(1,m+s+1)} + 2(2m-2s-1+l)\tau_{[0,l]}^{(1,m+s)}], \quad (4.5f)$$

where $z, l \in \{0, 1\}$, $m, s \geq 0$ and we set $\tilde{K}_{[l,1]}^{(-1)} = \tau_{[0,l]}^{(1,-1)} = 0$ once they appear on the r.h.s. of (4.5).

4.2 Symmetries for the isospectral IDNLS hierarchy

The algebraic structures in (3.15) also enable us to get two sets of symmetries of the isospectral IDNLS hierarchy (3.14a). For this we have the following theorem.

Theorem 4.2. *Each equation $i^{1-j}Q_{n,t_{k,j}} = \tilde{K}_{[j,1]}^{(k)}$ in the isospectral IDNLS hierarchy (3.14a) has two sets of symmetries. When $j = 0$ these symmetries are*

$$K\text{-symmetries: } \{i^{l-1}\tilde{K}_{[l,1]}^{(m)}\}, \quad l \in \{0, 1\}, \quad (4.6a)$$

$$\tau\text{-symmetries: } \tau_{[0,0]}^{(k,s)} = -2k t_{k,0}\tilde{K}_{[1,1]}^{(k+s-1)} - i\tilde{\sigma}_{[0,1]}^{(s)}, \quad (4.6b)$$

$$\tau_{[0,1]}^{(k,s)} = -\frac{1}{2}ik t_{k,0}(\tilde{K}_{[0,1]}^{(k+s+1)} + 4\tilde{K}_{[0,1]}^{(k+s)}) + \tilde{\sigma}_{[1,1]}^{(s)}; \quad (4.6c)$$

and when $j = 1$ the symmetries are

$$K\text{-symmetries: } \{i^{l-1}\tilde{K}_{[l,1]}^{(m)}\}, \quad l \in \{0, 1\}, \quad (4.7a)$$

$$\tau\text{-symmetries: } \tau_{[1,0]}^{(k,s)} = -\frac{1}{2}i(k+1)t_{k,1}\tilde{K}_{[0,1]}^{(k+s+1)} - i(2k+1)t_{k,1}\tilde{K}_{[0,1]}^{(k+s)} - i\tilde{\sigma}_{[0,1]}^{(s)}, \quad (4.7b)$$

$$\tau_{[1,1]}^{(k,s)} = \frac{1}{2}(k+1)t_{k,1}\tilde{K}_{[1,1]}^{(k+s+1)} + (2k+1)t_{k,1}\tilde{K}_{[1,1]}^{(k+s)} + \tilde{\sigma}_{[1,1]}^{(s)}. \quad (4.7c)$$

Symmetries for each equation can form a Lie algebra and structures are described as

$$\llbracket i^{z-1} \tilde{K}_{[z],1}^{(m)}, i^{l-1} \tilde{K}_{[l],1}^{(s)} \rrbracket_{Q_n} = 0, \quad (4.8a)$$

$$\llbracket -i \tilde{K}_{[0],1}^{(m)}, \tau_{[j,0]}^{(k,s)} \rrbracket_{Q_n} = -2m \tilde{K}_{[1],1}^{(m+s-1)}, \quad (4.8b)$$

$$\llbracket -i \tilde{K}_{[0],1}^{(m)}, \tau_{[j,1]}^{(k,s)} \rrbracket_{Q_n} = -\frac{1}{2} im (\tilde{K}_{[0],1}^{(m+s+1)} + 4 \tilde{K}_{[0],1}^{(m+s)}), \quad (4.8c)$$

$$\llbracket \tilde{K}_{[1],1}^{(m)}, \tau_{[j,l]}^{(k,s)} \rrbracket_{Q_n} = \frac{1}{2} i^{l-1} [(m+1) \tilde{K}_{[l],1}^{(m+s+1)} + 2(2m+1) \tilde{K}_{[l],1}^{(m+s)}], \quad (4.8d)$$

$$\llbracket \tau_{[j,0]}^{(k,m)}, \tau_{[j,0]}^{(k,s)} \rrbracket_{Q_n} = -2(m-s) \tau_{[j,1]}^{(k,m+s-1)}, \quad (4.8e)$$

$$\llbracket \tau_{[j,l]}^{(k,m)}, \tau_{[j,1]}^{(k,s)} \rrbracket_{Q_n} = \frac{1}{2} [(m-s-1+l) \tau_{[j,l]}^{(k,m+s+1)} + 2(2m-2s-1+l) \tau_{[j,l]}^{(k,m+s)}], \quad (4.8f)$$

where $j, z, l \in \{0, 1\}$, $k, m, s \geq 0$ and we set $\tilde{K}_{[l],1}^{(-1)} = \tau_{[j,l]}^{(k,-1)} = 0$ once they appear on the r.h.s. of (4.8). Especially, when $j = 0, k = 1$ the above results reduce to Theorem 4.1.

4.3 Relations between flows and the recursion operator \mathcal{L}

Theorem 4.3. The flows $\{K_{[j]}^{(m)}\}$ and $\{\sigma_{[j]}^{(m)}\}$ and their recursion operator \mathcal{L} satisfy

$$\mathcal{L}'[K_{[j]}^{(m)}] - [K_{[j]}^{(m)'}, \mathcal{L}] = 0, \quad j \in \{0, 1\}, \quad (4.9a)$$

$$\mathcal{L}'[\sigma_{[0]}^{(m)}] - [\sigma_{[0]}^{(m)'}, \mathcal{L}] - \mathcal{L}^m(L - L^{-1}) = 0, \quad (4.9b)$$

$$\mathcal{L}'[\sigma_{[1]}^{(m)}] - [\sigma_{[1]}^{(m)'}, \mathcal{L}] - \frac{1}{2} \mathcal{L}^{m+2} - 2\mathcal{L}^{m+1} = 0, \quad (4.9c)$$

where $m = 0, 1, 2, \dots$.

Proof. We only prove (4.9c). The other two can be proved similarly. We start from the relation

$$L'[\sigma^{(m)}] - [\sigma^{(m)'}, L] - 2L^{m+1} = 0, \quad (4.10a)$$

$$(L^{-1})'[\sigma^{(m)}] - [\sigma^{(m)'}, L^{-1}] + 2L^{m-1} = 0, \quad (4.10b)$$

in which (4.10a) was given in Ref. [7] and (4.10b) can be proved similarly but we here skip the proof. We can express $\mathcal{L}'[\sigma_{[1]}^{(m)}] - [\sigma_{[1]}^{(m)'}, \mathcal{L}]$ in terms of L, L^{-1} and $\sigma^{(s)}$ and then making use of the above relation we find

$$\begin{aligned} & \mathcal{L}'[\sigma_{[1]}^{(m)}] - [\sigma_{[1]}^{(m)'}, \mathcal{L}] \\ &= \frac{1}{2} \sum_{h=0}^m \sum_{k=0}^{m-h} C_m^h C_{m-h}^k (-2)^k (L^{m-2h-k+2} - L^{m-2h-k} - L^{m-2h-k} + (L^{-1})^{m-2h-k-2}) \\ &= \frac{1}{2} (L^2 \mathcal{L}^m - \mathcal{L}^m - \mathcal{L}^m + L^{-2} \mathcal{L}^m) \\ &= \frac{1}{2} \mathcal{L}^{m+2} + 2\mathcal{L}^{m+1}. \end{aligned}$$

Thus we complete the proof. \square

We note that these relations (4.9) can be employed to prove the Lemma 3.1 if we use inductive approach (cf. [10, 11] for continuous cases).

5 Continuous limit

5.1 DAKNS flows

In Sec.3.2 we introduced flows $\{K_{[j]}^{(l)}\}$ and $\{\sigma_{[j]}^{(l)}\}$ given by (3.16), which were referred to as DAKNS flows. In fact, in continuous limit these flows just go to the continuous AKNS isospectral and non-isospectral flows.

Let us consider the following limit (cf. [4]):

- replacing Q_n and R_n with hq_n and hr_n , where h is the real step parameter (the lattice spacing),
- $n \rightarrow \infty$, $h \rightarrow 0$ such that nh finite,
- introducing continuous variable $x = x_0 + nh$, then for a scalar function, for example, q_n , one has $q_{n+j} = q(x + jh)$. For convenience we take $x_0 = 0$.

Following the above limit procedure, one can find that

$$K_{[0]}^{(0)} \rightarrow \begin{pmatrix} q \\ -r \end{pmatrix} \quad (5.1)$$

and the leading term is of $O(h)$;

$$K_{[1]}^{(0)} \rightarrow \begin{pmatrix} q \\ r \end{pmatrix}_x \quad (5.2)$$

and the leading term is of $O(h^2)$. Besides, for any given scalar functions $f_{1,n}$ and $f_{2,n}$, applying the above continuous limit on the operator \mathcal{L} (3.3) and defining integrate operator $\partial_x^{-1} \sim h(E-1)^{-1}$, one can find that the leading term is of $O(h^2)$, which gives

$$\begin{aligned} \mathcal{L} \begin{pmatrix} f_{1,n} \\ f_{2,n} \end{pmatrix} &\rightarrow \left(\partial_x^2 - 4qr - 2 \begin{pmatrix} q_x \\ r_x \end{pmatrix} \partial_x^{-1}(r, q) - 2 \begin{pmatrix} q \\ -r \end{pmatrix} \partial_x^{-1}(-r_x, q_x) \right) \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \\ &= L_{AKNS}^2 \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}, \end{aligned} \quad (5.3)$$

where L_{AKNS} is the well known recursion operator of the AKNS system, defined as (cf. [10, 12])

$$L_{AKNS} = -\sigma_3 \partial_x + 2\sigma_3 \begin{pmatrix} q \\ r \end{pmatrix} \partial_x^{-1}(r, q), \quad (5.4)$$

in which σ_3 is the Pauli matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. This result means in continuous limit \mathcal{L} goes to the square of the AKNS recursion operator. Then, noting that (5.1) and (5.2) are nothing but the first two flows in the AKNS hierarchy (cf. [10, 13]), now it is clear that in continuous limit

$$K_{[0]}^{(m)} \rightarrow K_{AKNS}^{(2m)}, \quad \text{leading term } O(h^{2m+1}), \quad (5.5a)$$

$$K_{[1]}^{(m)} \rightarrow K_{AKNS}^{(2m+1)}, \quad \text{leading term } O(h^{2m+2}), \quad (5.5b)$$

where $\{K_{AKNS}^{(s)}\}$ are the hierarchy of the AKNS isospectral flows.

After similar discussions, for the non-isospectral flows $\{\sigma_{[j]}^{(m)}\}$ we have

$$\sigma_{[0]}^{(0)} \rightarrow \begin{pmatrix} xq \\ -xr \end{pmatrix}, \quad \text{leading term } O(1), \quad (5.6a)$$

$$\sigma_{[1]}^{(0)} \rightarrow \begin{pmatrix} xq \\ xr \end{pmatrix}_x, \quad \text{leading term } O(h), \quad (5.6b)$$

which are the first two AKNS non-isospectral flows (cf. [14]), and

$$\sigma_{[0]}^{(m)} \rightarrow \sigma_{AKNS}^{(2m)}, \quad \text{leading term } O(h^{2m}), \quad (5.7a)$$

$$\sigma_{[1]}^{(m)} \rightarrow \sigma_{AKNS}^{(2m+1)}, \quad \text{leading term } O(h^{2m+1}), \quad (5.7b)$$

where $\{\sigma_{AKNS}^{(s)}\}$ are the hierarchy of the AKNS non-isospectral flows.

5.2 Isospectral AKNS hierarchy and NLS hierarchy

Based on the above discussion on continuous limit, we can define the DAKNS hierarchy as

$$u_{n,t_{m,j}} = K_{[j]}^{(m)}, \quad j \in \{0, 1\}, \quad m = 0, 1, \dots, \quad (5.8)$$

which is an isospectral evolution equation hierarchy. Consider continuous limit of the above hierarchy. The dominated terms on both sides should have same order in terms of h . Since we have replaced u_n by $h \cdot (q, r)^T$, we still need to replace $t_{m,j}$ by $t_{2m+j} \cdot h^{-(2m+j)}$, i.e.,

$$t_{m,j} \rightarrow t_{2m+j} \cdot h^{-(2m+j)}, \quad (5.9)$$

so that the left side of (5.8) is $h^{2m+1+j} \cdot \mathcal{U}_{t_{2m+j}}$, i.e., of $O(h^{2m+1+j})$, where $\mathcal{U} = (q, r)^T$. Thus in continuous limit the DAKNS hierarchy (5.8) goes to the ANKS isospectral evolution equation hierarchy

$$\mathcal{U}_{t_{2m+j}} = K_{AKNS}^{(2m+j)}, \quad j \in \{0, 1\}, \quad m = 0, 1, \dots. \quad (5.10)$$

Then we define

$$\tilde{K}_{AKNS}^{(2m+j)} = K_{AKNS}^{(2m+j)}|_{r=-\varepsilon q^*}, \quad j \in \{0, 1\}, \quad m = 0, 1, \dots. \quad (5.11)$$

Its first element $\tilde{K}_{AKNS,1}^{(2m+j)}$ generates the NLS hierarchy

$$i^{1-j} q_{t_{2m+j}} = \tilde{K}_{AKNS,1}^{(2m+j)}, \quad j \in \{0, 1\}, \quad m = 0, 1, \dots, \quad (5.12)$$

which is just the continuous limit of the IDNLS hierarchy (3.14a).

5.3 Symmetries

Using the algebra \mathcal{D} defined by (3.17) one can construct two sets of symmetries for any equation

$$u_{n,t_{l,j}} = K_{[j]}^{(l)} \quad (5.13)$$

in the DAKNS hierarchy (5.8). When $j = 0$ these symmetries are

$$K\text{-symmetries: } K_{[k]}^{(m)}, \quad k \in \{0, 1\}, \quad (5.14a)$$

$$\tau\text{-symmetries: } \mathcal{T}_{[0,0]}^{(l,s)} = 2lt_{l,0}K_{[1]}^{(l+s-1)} + \sigma_{[0]}^{(s)}, \quad (5.14b)$$

$$\mathcal{T}_{[0,1]}^{(l,s)} = \frac{1}{2}lt_{l,0}K_{[0]}^{(l+s+1)} + 2lt_{l,0}K_{[0]}^{(l+s)} + \sigma_{[1]}^{(s)}; \quad (5.14c)$$

and when $j = 1$ the symmetries are

$$K\text{-symmetries: } K_{[k]}^{(m)}, \quad k \in \{0, 1\}, \quad (5.15a)$$

$$\tau\text{-symmetries: } \mathcal{T}_{[1,0]}^{(l,s)} = \frac{1}{2}(l+1)t_{l,1}K_{[0]}^{(l+s+1)} + (2l+1)t_{l,1}K_{[0]}^{(l+s)} + \sigma_{[0]}^{(s)}, \quad (5.15b)$$

$$\mathcal{T}_{[1,1]}^{(l,s)} = \frac{1}{2}(l+1)t_{l,1}K_{[1]}^{(l+s+1)} + (2l+1)t_{l,1}K_{[1]}^{(l+s)} + \sigma_{[1]}^{(s)}. \quad (5.15c)$$

The symmetries for (5.13) form a Lie algebra with structure

$$[[K_{[z]}^{(m)}, K_{[k]}^{(s)}]] = 0, \quad (5.16a)$$

$$[[K_{[0]}^{(m)}, \mathcal{T}_{[j,0]}^{(l,s)}]] = 2mK_{[1]}^{(m+s-1)}, \quad (5.16b)$$

$$[[K_{[0]}^{(m)}, \mathcal{T}_{[j,1]}^{(l,s)}]] = \frac{1}{2}m(K_{[0]}^{(m+s+1)} + 4K_{[0]}^{(m+s)}), \quad (5.16c)$$

$$[[K_{[1]}^{(m)}, \mathcal{T}_{[j,k]}^{(l,s)}]] = \frac{1}{2}(m+1)K_{[k]}^{(m+s+1)} + (2m+1)K_{[k]}^{(m+s)}, \quad (5.16d)$$

$$[[\mathcal{T}_{[j,0]}^{(l,m)}, \mathcal{T}_{[j,0]}^{(l,s)}]] = 2(m-s)\mathcal{T}_{[j,1]}^{(l,m+s-1)}, \quad (5.16e)$$

$$[[\mathcal{T}_{[j,k]}^{(l,m)}, \mathcal{T}_{[j,1]}^{(l,s)}]] = \frac{1}{2}(m-s-1+k)\mathcal{T}_{[j,k]}^{(l,m+s+1)} + (2m-2s-1+k)\mathcal{T}_{[j,k]}^{(l,m+s)}, \quad (5.16f)$$

where $j, k, z \in \{0, 1\}$, $l, m, s \geq 0$ and we set $K_{[z]}^{(-1)} = \mathcal{T}_{[j,k]}^{(l,-1)} = 0$ once they appear on the r.h.s. of (5.16).

For the continuous flows $\{K_{AKNS}^{(m)}\}$ and $\{\sigma_{AKNS}^{(s)}\}$, it has been known that (cf. [10, 15–17]) they form an algebra as well. We denote this algebra by \mathcal{C} . Its structure is

$$[[K_{AKNS}^{(m)}, K_{AKNS}^{(s)}]] = 0, \quad (5.17a)$$

$$[[K_{AKNS}^{(m)}, \sigma_{AKNS}^{(s)}]] = mK_{AKNS}^{(m+s-1)}, \quad (5.17b)$$

$$[[\sigma_{AKNS}^{(m)}, \sigma_{AKNS}^{(s)}]] = (m-s)\sigma_{AKNS}^{(m+s-1)}, \quad (5.17c)$$

where $m, s \geq 0$ and we set $K_{AKNS}^{(-1)} = \sigma_{AKNS}^{(-1)} = 0$ once they appear on the r.h.s. of (5.17). This algebra can be generated by

$$\{K_{AKNS}^{(1)}, \sigma_{AKNS}^{(0)}, \sigma_{AKNS}^{(3)}\}. \quad (5.18)$$

From \mathcal{C} one can have two sets of symmetries for each AKNS equation

$$\mathcal{U}_{t_l} = K_{AKNS}^{(l)} \quad (5.19)$$

in the hierarchy (5.10), the symmetries are

$$K\text{-symmetries: } K_{AKNS}^{(m)}, \quad (5.20a)$$

$$\tau\text{-symmetries: } \mathcal{T}_{AKNS}^{(l,s)} = lt_l K_{AKNS}^{(l+s-1)} + \sigma_{AKNS}^{(s)}, \quad (5.20b)$$

which form a Lie algebra with (cf. [10, 15–17])

$$[[K_{AKNS}^{(m)}, K_{AKNS}^{(s)}]] = 0, \quad (5.21a)$$

$$[[K_{AKNS}^{(m)}, \mathcal{T}_{AKNS}^{(l,s)}]] = m K_{AKNS}^{(m+s-1)}, \quad (5.21b)$$

$$[[\mathcal{T}_{AKNS}^{(l,m)}, \mathcal{T}_{AKNS}^{(l,s)}]] = (m-s) \mathcal{T}_{AKNS}^{(l,m+s-1)}, \quad (5.21c)$$

where $l, m, s \geq 0$ and we set $K_{AKNS}^{(-1)} = \mathcal{T}_{AKNS}^{(l,-1)} = 0$ once they appear on the r.h.s. of (5.21).

Starting from the relations (5.17) and employing similar discussions as we have done for the IDNLS hierarchy in Sec.4, one can have two sets of symmetries for the l th equation in the NLS hierarchy (5.12):

$$q_{t_l} = \mu_l \tilde{K}_{AKNS,1}^{(l)}, \quad \mu_l = \begin{cases} -i, & l \text{ is even,} \\ 1, & l \text{ is odd.} \end{cases} \quad (5.22)$$

The symmetries are (cf. [10])

$$K\text{-symmetries: } \mu_m \tilde{K}_{AKNS,1}^{(m)}, \quad (5.23a)$$

$$\tau\text{-symmetries: } \tilde{\mathcal{T}}^{(l,s)} = \mu_l \mu_s lt_l \tilde{K}_{AKNS,1}^{(l+s-1)} + \mu_s \tilde{\sigma}_{AKNS,1}^{(s)}, \quad (5.23b)$$

which compose a Lie algebra with structure

$$[[\mu_m \tilde{K}_{AKNS,1}^{(m)}, \mu_s \tilde{K}_{AKNS,1}^{(s)}]]_q = 0, \quad (5.24a)$$

$$[[\mu_m \tilde{K}_{AKNS,1}^{(m)}, \tilde{\mathcal{T}}^{(l,s)}]]_q = m \mu_m \mu_s \tilde{K}_{AKNS,1}^{(m+s-1)}, \quad (5.24b)$$

$$[[\tilde{\mathcal{T}}^{(l,m)}, \tilde{\mathcal{T}}^{(l,s)}]]_q = (m-s) \frac{\mu_m \mu_s}{\mu_{m+s-1}} \tilde{\mathcal{T}}^{(l,m+s-1)}, \quad (5.24c)$$

where $l, m, s \geq 0$ and we set $\tilde{K}_{AKNS,1}^{(-1)} = \tilde{\mathcal{T}}^{(l,-1)} = 0$ once they appear on the r.h.s. of (5.24). In the product $[[\cdot, \cdot]]_q$ the Gateaux derivative is defined w.r.t. q . When $l = 1$, they are reduced the symmetries and algebra for the NLS equation.

The symmetries (5.14), (5.15) and (5.20) are related together by continuous limit. Same relations hold for the symmetries in Theorem 4.2 and (5.23). We skip the detailed discussions for these connections.

6 Algebra deformations and understanding

We have presented symmetries and their algebras for the continuous AKNS and NLS hierarchies. Comparing them with those for the discrete cases, one can find not only the form of symmetries but also the structures of algebras are different. Since the algebras \mathcal{D} and \mathcal{C} (see (3.17) and (5.17)) play key roles for generating symmetries, let us focus on \mathcal{D} and \mathcal{C} and see the difference between them in the light of the correspondence (5.5) and (5.7). Several of these deformations from \mathcal{D} and \mathcal{C} are listed in the following:

- different structures and different generators;
- $\{K_{[0]}^{(0)}, K_{[1]}^{(0)}, K_{[0]}^{(1)}, \sigma_{[0]}^{(0)}\}, \{\sigma_{[0]}^{(0)}, \sigma_{[1]}^{(0)}, \sigma_{[0]}^{(1)}\}$ and $\{K_{AKNS}^{(0)}, K_{AKNS}^{(1)}, K_{AKNS}^{(2)}, \sigma_{AKNS}^{(0)}\}, \{\sigma_{AKNS}^{(0)}, \sigma_{AKNS}^{(1)}, \sigma_{AKNS}^{(2)}\}$ are subalgebras of \mathcal{D} and \mathcal{C} respectively but with different structures;
- $\{K_{AKNS}^{(0)}, K_{AKNS}^{(1)}, \sigma_{AKNS}^{(0)}, \sigma_{AKNS}^{(1)}\}$ is a subalgebra of \mathcal{C} , but for \mathcal{D} we do not find any similar subalgebras (containing at least two non-isospectral flows and one isospectral flow).

To understand these deformations, we introduce *degree* for flows $\{K_{[j]}^{(m)}\}$ and $\{\sigma_{[j]}^{(m)}\}$, variable $t_{m,j}$ and the recursion operator \mathcal{L} . For a function $f(n, h, t)$ (or an operator) by $\deg f$ we mean the order of h of the denominator term (or leading term) in continuous limit. So we can define

$$\deg K_{[j]}^{(m)} = 2m + 1 + j, \quad (6.1a)$$

$$\deg \sigma_{[j]}^{(m)} = 2m + j, \quad (6.1b)$$

$$\deg \mathcal{L} = 2, \quad (6.1c)$$

$$\deg t_{m,j} = -(2m + j), \quad (6.1d)$$

$$\deg u_n = 1, \quad (6.1e)$$

where (6.1d) is from (5.9). Thus, after taking continuous limit only the terms with the lowest degree are left while others disappear. We also note that due to (6.1e) and the definition of Gateaux derivative

$$\deg f(u_n)'[g(u_n)] = \deg \llbracket f(u_n), g(u_n) \rrbracket = \deg f(u_n) + \deg g(u_n) - \deg u_n. \quad (6.2)$$

Now let us take (3.17c) and (5.17b) as an example to see the role that the degrees play in continuous limit and understanding those deformations. (3.17c) reads

$$\llbracket K_{[0]}^{(m)}, \sigma_{[1]}^{(s)} \rrbracket = \frac{1}{2}m(K_{[0]}^{(m+s+1)} + 4K_{[0]}^{(m+s)}), \quad (6.3)$$

in which

$$\deg \llbracket K_{[0]}^{(m)}, \sigma_{[1]}^{(s)} \rrbracket = 2(m+s)+1, \quad \deg K_{[0]}^{(m+s+1)} = 2(m+s)+3, \quad \deg K_{[0]}^{(m+s)} = 2(m+s)+1. \quad (6.4)$$

With the help of *degree* and noting that the correspondence (5.5) and (5.7), after taking continuous limit, only those terms with the lowest degree are left and consequently (6.3) goes to

$$\llbracket K_{AKNS}^{(2m)}, \sigma_{AKNS}^{(2s+1)} \rrbracket = 2mK_{AKNS}^{(2(m+s))},$$

which belongs to relation (5.17b). Similar to this example we can examine degrees of other formulas in (3.17) and in this way we can explain why the algebra \mathcal{D} goes to \mathcal{C} in continuous limit although they have different structures.

Next we turn to τ -symmetries. Some of them contain different number of terms in discrete case and continuous case. To explain the difference we consider (5.14b), (5.14c), (5.15b), (5.15c) and (5.20b) as an example. In (5.14b)

$$\deg(t_{l,0}K_{[1]}^{(l+s-1)}) = -2l + [2(l+s-1) + 1 + 1] = 2s, \quad \deg \sigma_{[0]}^{(s)} = 2s,$$

which are same. Thus in continuous limit (5.14b) yields

$$\tau_{AKNS}^{(2l,2s)} = 2lt_{2l}K_{AKNS}^{(2(l+s)-1)} + \sigma_{AKNS}^{(2s)}.$$

In (5.14c) the degrees are

$$\deg(t_{l,0}K_{[0]}^{(l+s+1)}) = 2s + 3, \quad \deg(t_{l,0}K_{[0]}^{(l+s)}) = \deg \sigma_{[1]}^{(s)} = 2s + 1.$$

Thus in continuous limit the term $t_{l,0}K_{[0]}^{(l+s+1)}$ will disappear due to higher degree and then (5.14c) goes to

$$\tau_{AKNS}^{(2l,2s+1)} = 2lt_{2l}K_{AKNS}^{(2(l+s))} + \sigma_{AKNS}^{(2s+1)}.$$

Similarly, (5.15b) goes to

$$\tau_{AKNS}^{(2l+1,2s)} = (2l+1)t_{2l+1}K_{AKNS}^{(2(l+s))} + \sigma_{AKNS}^{(2s)},$$

and (5.15c) goes to

$$\tau_{AKNS}^{(2l+1,2s+1)} = (2l+1)t_{2l+1}K_{AKNS}^{(2(l+s)+1)} + \sigma_{AKNS}^{(2s+1)}.$$

The above four continuous limit results just compose the τ -symmetry (5.20b).

Now by means of *degree* we can understand the deformation of algebras and symmetries appearing in continuous limit. Finally, let us look at the relation (4.9). In continuous limit we have

$$(L_{AKNS}^2)'[K_{AKNS}^{(m)}] - [K_{AKNS}^{(m)'}, L_{AKNS}^2] = 0, \quad (6.5a)$$

$$(L_{AKNS}^2)'[\sigma_{AKNS}^{(m)}] - [\sigma_{AKNS}^{(m)'}, L_{AKNS}^2] - 2L_{AKNS}^{m+1} = 0. \quad (6.5b)$$

This is consistent with the result for the AKNS recursion operator and flows:

$$L_{AKNS}'[K_{AKNS}^{(m)}] - [K_{AKNS}^{(m)'}, L_{AKNS}] = 0, \quad (6.6a)$$

$$L_{AKNS}'[\sigma_{AKNS}^{(m)}] - [\sigma_{AKNS}^{(m)'}, L_{AKNS}] - L_{AKNS}^m = 0, \quad (6.6b)$$

where we have made use of the result

$$\frac{1}{2}(L - L^{-1}) \rightarrow L_{AKNS}, \quad \text{leading term : } O(h) \quad (6.7)$$

in continuous limit, which can be seen through a procedure similar to (5.3). Applying L_{AKNS} to (6.6) and using the Leibniz rule for two operators F and G : $(FG)'[h] = F'[h]G + F G'[h]$, one can derive (6.5) from (6.6). However, one may wonder that now that in continuous limit we have the operator $\frac{1}{2}(L - L^{-1}) \rightarrow L_{AKNS}$ but $\mathcal{L} \rightarrow L_{AKNS}^2$, why we use \mathcal{L} in stead of $\frac{1}{2}(L - L^{-1})$ to generate DAKNS hierarchy? In fact, it is true that different discrete flows can go to the same in continuous limit. Applying $\frac{1}{2}(L - L^{-1})$ on $K_{[0]}^{(0)}$ twice and taking $Q_n = -\varepsilon R_n^*$ we have a second IDNLS equation,

$$\begin{aligned} iQ_{n,t_2} = & \frac{1}{4}(1 + \varepsilon Q_n Q_n^*)[Q_{n+2}(1 + \varepsilon Q_{n+1} Q_{n+1}^*) + \varepsilon Q_n Q_{n+1} Q_{n-1}^* + \varepsilon Q_{n+1}^2 Q_n^* \\ & + Q_{n-2}(1 + \varepsilon Q_{n-1} Q_{n-1}^*) + \varepsilon Q_{n-1}^2 Q_n^* + \varepsilon Q_{n-1} Q_n Q_{n+1}^*] - \frac{1}{2}Q_n. \end{aligned} \quad (6.8)$$

It looks more complicated than the IDNLS equation (1.1) but it does go to the continuous NLS equation. Therefore we prefer to \mathcal{L} although it corresponds to L_{AKNS}^2 .

7 Conclusions

One of the main results of the paper is that we got infinitely many symmetries for the IDNLS equation (1.1) and the IDNLS hierarchy (3.14a). This was done through constructing the recursion operator $\tilde{\mathcal{L}}$, isospectral and non-isospectral IDNLS flows in scalar form and their algebraic structures (3.15).

A second result is on the DAKNS flows. These flows are generated by the basic flows $K_{[0]}^{(0)}, K_{[1]}^{(0)}, \sigma_{[0]}^{(0)}, \sigma_{[1]}^{(0)}$ and the recursion operator \mathcal{L} . By continuous limit one can build direct correspondence between these flows and the continuous AKNS isospectral and non-isospectral flows. Meanwhile, \mathcal{L} goes to the square of the AKNS recursion operator L_{AKNS} in the same continuous limit procedure. These DAKNS flows form a Lie algebra \mathcal{D} of which the structures (3.17) are derived from the basic algebraic relations (2.4) of those two-potential AL flows. It has been shown that this algebra \mathcal{D} plays a key role in constructing symmetries and their algebraic structures for both the IDNLS hierarchy and DAKNS hierarchy.

The final main result is on the algebra deformations and explanations. We listed out some deformations of algebras when they go to continuous case. In fact, in Ref. [5] a contraction of subalgebras has been reported, but in that case the correspondence between discrete case and continuous case is not direct and some linear combinations were involved. As we can see in the present paper the correspondence between the discrete and continuous AKNS flows is direct (see (5.5) and (5.7)); and by means of continuous limit and the lattice spacing parameter h we introduced *degree* for discrete elements, as listed in (6.1). Calculating the degree of each term one can understand the algebra deformations before and after taking continuous limit.

Finally, we note that \mathcal{D} is not a centerless KMV algebra, but it is somehow related to this type. On one hand, \mathcal{D} is derived from the centerless KMV algebra (2.4). On the other hand, \mathcal{D} and \mathcal{C} are a continuum in continuous limit and \mathcal{C} is a centerless KMV algebra. It is not rare to see the algebraic structure changes in discrete cases. For example, besides the subalgebra contraction found in [5] and the algebra deformations listed in this paper, the symmetry algebra of the differential-difference KP equation also has a non-centreless Kac-Moody-Virasoro structure [18]. We believe continuous limit and *degree* are good means to understand these changes.

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